Integration on Computer Algebra Systems

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Abstract

In this article, we consider ten indefinite integrals and the ability of three computer algebra systems (CAS) to evaluate them in closed-form, appealing only to the class of real, elementary functions. Although these systems have been widely available for many years and have undergone major enhancements in new versions, it is interesting to note that there are still indefinite integrals that escape the capacity of these systems to provide antiderivatives. When this occurs, we consider what a user may do to find a solution with the aid of a CAS.

1. Introduction

We will explore the use of three CAS's in the evaluation of indefinite integrals: Maple 11, Mathematica 6.0.2 and the Texas Instruments (TI) 89 Titanium graphics calculator. We consider integrals of real elementary functions of a single real variable in the examples that follow. Students often believe that a good CAS will enable them to solve any problem when there is a known solution; these examples are useful in helping instructors show their students that this is not always the case, even in a calculus course. A CAS may provide a solution, but in a form containing special functions unfamiliar to calculus students, or too cumbersome for students to use directly, [1].

Students may ask, "Why do we need to learn integration methods when our CAS will do all the exercises in the homework?" As instructors, we want our students to come away from their mathematics experience with some capacity to make intelligent use of a CAS when needed. Few of us know precisely what algorithms our CAS uses in determining antiderivatives, or why the CAS may fail to provide an antiderivative on a particular example; however, good facility in hand-computation together with a CAS enables users to handle more sophisticated examples whenever they arise.

All three systems do very well in providing antiderivatives to the standard forms in widely available tables of integration, [2]. The integrands we consider here are esoteric in nature, to perform a "stress test" of the integration capabilities of these systems. What follows are slightly modified quotes from technical specialists at Waterloo Maple, Wolfram Research, and Texas Instruments as to specifics of their integration algorithms.

Maple has various integrators that are used to integrate different types of functions. This includes general-purpose integrators such as a Risch integrator for elementary functions, and the MeijerG integrator. (A Risch algorithm uses induction on the number of monomial and algebraic extensions required to construct the function field containing the integrand beginning with the rational functions as base field, [3]. Products of MeijerG functions are relatively easy to integrate symbolically.) There are a number of specialized integrators for handling elliptic integrals, integrals of algebraic functions, and integrals involving special functions, etc. This set of specialized integrators can be extended by the user, who may write, for example, a procedure `int/F' for integrating expressions involving F(x). This procedure will be automatically used by the command "int."

Maple first tries to simplify the problem using various transformations, substitutions, etc. After that Maple tries to determine if any of the special cases is applicable and uses the corresponding integrator. If this does not work, Maple tries the general-purpose integrators. Note that some of the integrators may return partial answers in which case Maple will try to complete integration using other methods, [4]. Maple's "int" command uses approximately 60,000 lines of code (including definite integration).

Mathematica 6.0.2 (produced by Wolfram Research) first looks for certain simple cases to do by substitution methods. Most such cases involve trigonometric or hyperbolic functions, possibly multiplied by polynomials or exponentials. These cases cannot be evaluated effectively by general algorithms. Mathematica next applies Risch code that may separate the integrand into summands. If some or all of those cannot be integrated by the Risch code, it applies various special function methods. There is dedicated code, for example, to handle elliptic integrals and algebraic functions or special functions where specific algorithms (based on the theory of Hypergeometric PFQ and MeijerG functions) are used, [5].

For indefinite integrals, an extended version of the Risch code is used whenever both the integrand and integral can be expressed in terms of elementary functions, exponential integral functions, polylogarithms and other related functions. For other indefinite integrals, heuristic simplification followed by pattern matching is used. The algorithms in Mathematica cover all of the indefinite integrals in standard reference books such as [6]. The command "Integrate" uses about 25,000 lines of Mathematica code and about 30,000 lines of C code.

On the TI-89 graphics calculator, antiderivatives are determined by substitutions, integration by parts, and partial-fraction expansion, much as described in [3]. Antiderivatives are not computed by any of the Risch-type algorithms. Definite integrals are determined by subdividing the interval at detected singularities, then for each interval, computing the difference of the limit of an antiderivative at the upper and lower integration limits. Except in exact mode, nINT() is used were applicable when symbolic methods do not succeed.

2. Integration Problems

For the ten integration problems included in this article, the following table indicates a success by a given CAS in terms of real functions on a particular problem with a check mark. The problems were selected from a group of more than 50 integrals tested on the three CAS; those selected cover a range of combinations of transcendental and algebraic functions and pose some difficulties to one or more of the CAS tested. For a selection of other integrals tested, please see the Appendix. These integrals are typically just beyond the scope of "Methods of Integration" chapters in Calculus texts, but most are within the ability of a strong calculus student to solve successfully. Students planning to continue in mathematics beyond calculus need to know that, while a good CAS is an indispensable tool, they still need to know the basics of computation by hand to enable effective use of a CAS. Many problems they will encounter in mathematics and the natural sciences do not lend themselves to simply inputting a few brief command lines into a CAS to obtain a result. Each problem that follows contains some details required for hand-computation to produce a new, simpler integral that is then within the capabilities of each of the three CAS.

Successes	**#1**	**#2**	**#3**	**#4**	**#5**	**#6**	**#7**	**#8**	**#9**	**#10**
MMA 6.0.2	✓	✓	✓	✓						✓
Maple 11	✓	✓	✓	✓		✓	✓	✓		
TI-89	✓	✓				✓			✓	

Problem #1. Evaluate $\int \arcsin(x)\ln(x)dx.$

This is a classic integration by parts problem; most students approached it by differentiating arcsin(x) and integrating ln(x) to obtain the integral

$$\int \frac{x\ln(x)dx}{\sqrt{1-x^2}}$$

using the "parts" formula. Maple 5 could not evaluate this second integral, but Maple 8 was able to. Maple 8 could not integrate the original integral, but Maple 11 now evaluates it correctly. Mathematica 6.0.2 and the TI-89 are able to give the solution to the original integral as

$$-\ln(\sqrt{1-x^2}-1)+\ln(x)+(x\arcsin(x)+\sqrt{1-x^2})\ln(x)-x\arcsin(x)-2\sqrt{1-x^2}+C.$$

This integral arose in a search for integration challenges since the integrand is the product of the function inverses to e^x and $\sin(x)$. If a user is unable to evaluate the original integral on their CAS, facility in integration by parts then enables the user to find a new second integral that is then handled by the CAS.

Problem #2. Evaluate $\int \frac{x\arcsin(x)dx}{\sqrt{1-x^2}}$.

Integration by parts, letting

$$u = \arcsin(x), dv = \frac{xdx}{\sqrt{1-x^2}}$$

quickly yields the solution

$$-\arcsin(x)\sqrt{1-x^2} + x + C\,.$$

Mathematica 6.0.2, Maple 11 and the TI-89 all gave this solution. Interestingly, this is one of several examples of a specific type of integrand that had a tendency to foil Maple's and Mathematica's integration algorithms in earlier versions. The integrand tends to contain three (or more) factors and requires integration by parts. One of the factors will be an inverse trigonometric or logarithmic function that we would differentiate in the "parts" process, and the other will typically be a product containing algebraic functions that we integrate in the "parts" process. This integration typically requires a change of variable (u-substitution) to obtain the "v" term needed to obtain the simpler integral. The new integrand will be algebraic, since the derivatives of the inverse trigonometric functions and the logarithms we consider are algebraic. If a CAS fails to give a solution here, the trigonometric substitution $x = \sin(\theta)$ converts the original integral to $\int \theta \sin(\theta) d\theta$, one that students should quickly recognize.

Problem #3. Evaluate $\int \arcsin(\sqrt{x+1} - \sqrt{x})dx$.

Using rationalizing substitution $x = \tan^2(\theta)$ to eliminate square roots, we have

$$2\int \arcsin(\sec\theta - \tan\theta)\tan\theta\sec^2\theta d\theta\,,$$

and rewriting the “arcsin” as an “arctan” form, we obtain the new integral

$$2\int \arctan\left(\sqrt{\frac{1}{2}}(\csc\theta - 1)\right)\tan(\theta)\sec^2\theta d\theta$$

Integration by parts now gives us

$$\tan^2\theta \arctan\left(\sqrt{\frac{1}{2}}(\csc\theta - 1)\right) + \frac{1}{\sqrt{2}}\int \frac{\tan\theta\csc\theta}{(\csc\theta + 1)\sqrt{\csc\theta - 1}} d\theta$$

The further rationalizing substitution $\csc\theta = \sec^2 u$ now gives us the rational trigonometric form

$$\sqrt{2}\int \frac{\sec^2 u du}{(\tan^2 u + 2)^2 \tan^2 u},$$

manageable by a CAS. The above is excerpted from the solution to this integral provided by J. Kingman, student; the final result is

$$(x+\frac{3}{8})\arcsin(\sqrt{x+1}-\sqrt{x})+\frac{\sqrt{2}}{4}\sqrt{\sqrt{x^2+x}+x}+\frac{\sqrt{2}}{8}\sqrt{\sqrt{x}(\sqrt{x+1}-\sqrt{x})^3}+C.$$

Mathematica 6.0.2 provided a more complicated solution here, valid over the complex domain. Maple 11 provided a solution containing tan(x/2), while the TI-89 gave no result. This example points up the real need for human-computer interaction to provide a result, when neither is able to handle this task easily alone!

Problem #4. Evaluate $\int \ln(1+x\sqrt{1+x^2})dx.$

This unusual integrand is a variation on the logarithmic form of $\arcsin h(x)$, and requires integration by parts:

$$u = \ln(1+x\sqrt{1+x^2}), \qquad dv = dx$$

$$du = \frac{\sqrt{1+x^2}+\dfrac{x^2}{\sqrt{1+x^2}}}{1+x\sqrt{1+x^2}}, \qquad v = x$$

On the new integral created by the "parts" formula, we make the rationalizing substitution $x = \tan\theta$ to obtain

$$2\int \frac{\sin\theta d\theta}{\cos^2\theta(\sin^2\theta-\sin\theta-1)}+\int \frac{\sin\theta d\theta}{1+\sin\theta-\sin^2\theta},$$

The rational trigonometric function of "θ" suggested here should be handled by any CAS after the application of the "parts" formula. Mathematica 6.0.2 used "$\tanh^{-1}$" to express the solution successfully; Maple 11 gave a more complicated result than that shown below. To simplify the final form of the solution, we set $a = \frac{\sqrt{5}+1}{2}$, and $t = \frac{\sqrt{x^2+1}-1}{x}$ to obtain

$$x\ln(1+x\sqrt{1+x^2})-2x+2\sqrt{a}\arctan(\sqrt{a}t-\frac{1}{\sqrt{a}})+\frac{1}{\sqrt{a}}\ln\left|\frac{t+a-\sqrt{a}}{t+a+\sqrt{a}}\right|+C.$$

Problem #5. Evaluate $\int\frac{\cos^2 x dx}{\sqrt{\cos^4 x+\cos^2 x+1}}$.

It might appear at first that this is an elliptic integral; however, if we multiply and divide by $\sin(x)$, and then make the judicious substitution $u=\cos^3(x)$, we obtain the new integral

$$\int\frac{\sin\theta\cos^2\theta d\theta}{\sqrt{1-\cos^6\theta}} = -\frac{1}{3}\arcsin(\cos^3(x))+C$$

If a definite integral is desired, one must use the fact that

$$|\sin(x)|=\sqrt{1-\cos^2(x)}$$

to obtain the correct sign. Mathematica 6.0.2 and Maple 11 both appealed to the use of elliptic functions (F and Pi) over the complex domain, while the TI-89 gave no result. The elliptic integrals of the first and second kind are special cases of the hypergeometric functions, [7]. It is reasonable to expect that a CAS "simplify" algorithm will often be unable to express an antiderivative using only real elementary functions when such exist, resulting from the complexity of the parameters involved in general hypergeometric functions. Since a typical integral similar to the one here will possess a non-elementary antiderivative, it is a difficult programming task to separate out those special cases where real, elementary antiderivatives exist.

Problem #6. Evaluate $\int\tan(x)\sqrt{1+\tan^4(x)}dx.$

If one multiplies and divides the integrand by $\sec^2(x)$ and makes use of the trigonometric identity

$$\tan^4(x)+1=\sec^4(x)-2\sec^2(x)+2,$$

the substitution $z=\sec^2(x)$ yields the integral

$$\frac{1}{2}\int\frac{\sqrt{z^2-2z+2}dz}{z}$$

that may be rationalized with a trigonometric substitution. The integrand containing the aforementioned function of "z" is now manageable on any of the three CAS tested. We obtain:

$$\frac{1}{2}\sqrt{1+\tan^4(x)}+\sqrt{2}\ln|\sec(x)|-\frac{1}{2}\ln|\sqrt{1+\tan^4(x)}+\tan^2(x)|-\frac{1}{2}\ln|\sqrt{2+2\tan^2(x)}-\tan^2(x)+1|+C.$$

The TI-89 gave this solution in terms of sine and cosine, Maple 11 gave a similar result containing "$\tanh^{-1}$" but Mathematica 6.0.2 gave a solution in terms of elliptic F and Pi functions.

Problem #7. Evaluate $\int\frac{\tan(x)dx}{\sqrt{\sec^3(x)+1}}$.

When we multiply and divide the integrand by $\sec(x)$, the substitution $u=\sec(x)$ produces the integral of the algebraic function

$$\int\frac{du}{u\sqrt{u^3+1}}$$

that is then easily rationalized by letting $v^2=u^3+1$, then integrated to give the final result

$$\frac{1}{3}\ln\left|\frac{\sqrt{\sec^3(x)+1}-1}{\sqrt{\sec^3(x)+1}+1}\right|+C.$$

Maple 11 gave the "$\tanh^{-1}$" form of the above logarithm. Both Mathematica 6.0.2 and the TI-89 did successfully integrate the algebraic function of "u" described above; however, the TI-89 gave no result for the original integral, and Mathematica 6.0.2 gave a solution using complex elliptic F and Pi functions. The ability of a CAS to integrate a trigonometric integrand of this type depends heavily upon its algorithm(s) used to convert the integrand to an algebraic form.

Problem #8. Evaluate $\int\sqrt{\tan^2(x)+2\tan(x)+2}\,dx.$

This integrand is recognized as the arc length formulation for the graph of the function $F(x)=x+\ln|\sec(x)|$. As with other complicated integrands containing square roots, we must eventually convert to a rational form. If we multiply and divide by the integrand, we get

$$\int\frac{\sec^2 x\,dx}{\sqrt{(\tan x+1)^2+1}}+\int\frac{(2\tan x+1)dx}{\sqrt{\tan^2 x+2\tan x+2}}$$

The first integral is easily integrated by the substitution $z=\tan(x)+1$. In the second integral, multiply and divide by $\sec^2(x)$. Substitute $\tan(\theta)=\tan(x)+1$, and make the further substitution $z=\tan(\theta/2)$ to produce the rational integrand

$$\frac{(z^2+4z-1)dz}{z^4+2z^3-2z+1}.$$

The denominator has two quadratic factors, which are

$$z^2+(1+\sqrt{2+\sqrt{5}})z+\frac{1}{2}(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}),\, and,\, z^2+(1-\sqrt{2+\sqrt{5}})z+\frac{1}{2}(1+\sqrt{5}-\sqrt{2+2\sqrt{5}}).$$

After using a partial fraction decomposition, we may express the solution (by hand) using

$$z=\frac{\sqrt{\tan^2(x)+2\tan(x)+2}-1}{\tan(x)+1}:$$

$$\ln|\tan(x)+1+\sqrt{\tan^2(x)+2\tan(x)+2}|+\sqrt{\frac{\sqrt{5}+1}{2}}\arctan\left(\frac{2z+1-\sqrt{2+\sqrt{5}}}{1-\sqrt{\sqrt{5}-2}}\right)$$

$$-\sqrt{\frac{\sqrt{5}+1}{2}}\arctan\left(\frac{2z+1+\sqrt{2+\sqrt{5}}}{1+\sqrt{\sqrt{5}-2}}\right)+\sqrt{\frac{\sqrt{5}-1}{8}}\ln\left|\frac{z^2+(1-\sqrt{2+\sqrt{5}})z+\frac{1}{2}(1+\sqrt{5}-\sqrt{2+2\sqrt{5}})}{z^2+(1+\sqrt{2+\sqrt{5}})z+\frac{1}{2}(1+\sqrt{5}+\sqrt{2+2\sqrt{5}})}\right|+C.$$

Here, the TI-89 converted the original integral to one involving sine and cosine but gave no result, Mathematica 6.0.2 gave a messy solution using “Root” and complex elliptic F and Pi functions, and Maple 11 gave the solution in terms of real functions, though somewhat less simplified than the above result.

Problem #9. Evaluate $\int \sin(x)\arctan(\sqrt{\sec(x)-1})dx.$

This integral is curious in that its integrand contains a product of both a trigonometric and an inverse trigonometric function, an unusual combination. One would naturally attempt to solve this by integration by parts, differentiating the "arctan" form and integrating $\sin(x)$:

$$-\cos x\arctan(\sqrt{\sec x-1})+\frac{1}{2}\int\frac{\sin x dx}{\sqrt{\sec x-1}}$$

Now, rationalize the denominator in the new integrand by setting $\sec(x)=\sec^2(u)$ and use appropriate identities. The final result is

$$-\cos(x)\arctan(\sqrt{\sec(x)-1})+\frac{1}{2}arc\sec(\sqrt{\sec(x)})+\frac{\sqrt{\sec(x)-1}}{2\sec(x)}+C.$$

The TI-89 gave the following solution,

$$\frac{-1}{2}\left((2\cos(x)-1)\tan^{-1}\left(\sqrt{\frac{-(\cos(x)-1)}{\cos(x)}}\right)-\cos(x)\sqrt{\frac{-(\cos(x)-1)}{\cos(x)}}\right),$$

and Mathematica 6.0.2 gave a solution using elliptic F and Pi functions. Maple 11 was unable to give any solution here. A user encountering difficulties with this example should integrate by parts as suggested, yielding the new integral as above; this should be evaluated by any CAS.

Problem #10. Evaluate $\int\frac{x^3 e^{\arcsin(x)}dx}{\sqrt{1-x^2}}$.

In this integral, it is clear that it will eventually require integration by parts, even if we made a substitution first. Letting $x=\sin(u)$, our new integral is:

$$\int e^u \sin^3 u du$$

Using the identity

$$\sin^3(u)=\frac{3}{4}\sin(u)-\frac{1}{4}\sin(3u)$$

and integrating by parts twice, we find the solution

$$\frac{e^{\arcsin(x)}}{10}(x^3+3x-3(x^2+1)\sqrt{1-x^2})+C.$$

Mathematica 6.0.2 first gave a version of this solution using sine and cosine of 3arcsin(x), but simplified this is another step; neither Maple 11 nor the TI-89 gave a result. A CAS user may rewrite the original integral with the above identity and allow the CAS to carry out the integration by parts to find the solution. Interestingly, Maple 11 and the TI-89 are unable to evaluate

$$\int e^{\arcsin(x)}dx,$$

whose integrand is the function inverse to $\sin(\ln(x))$ (with x restricted to the interval $[e^{-\pi/2}, e^{\pi/2}]$). The latter is an integrand found in the "integration by parts" exercises in many beginning calculus texts, [8]. In this latter integral, the substitution $x = \sin(\theta)$ yields the standard integral $\int e^{\theta}\cos(\theta)d\theta$, easily recognized by students.

3. Conclusion

It is important to reiterate that the three CAS tested here are very powerful, efficient systems that will perform the vast majority of integration work required in applications. One may write symbolic programs within these systems to handle any esoteric integrals one might encounter. In practice, any CAS will provide a solution to a problem in definite integration when that is all that is required, especially when a closed-form antiderivative is cumbersome. Integration provides students with their first example of an operation that is sufficiently difficult algorithmically that they cannot always expect to find solutions in a convenient form using a CAS; this emphasizes the need for students to achieve facility in computation by hand, to more effectively use a CAS when required.
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4. References

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Appendix Other Integrals Tested

$\int \frac{x\ln(1+x^2)\ln(x+\sqrt{1+x^2})dx}{\sqrt{1+x^2}}$ $\int \arctan(x+\sqrt{1-x^2})dx$ $\int \frac{x\arctan(x+\sqrt{1-x^2})dx}{\sqrt{1-x^2}}$

$\int \frac{\arcsin x dx}{1+\sqrt{1-x^2}}$ $\int \frac{\ln(x+\sqrt{1+x^2})dx}{(1-x^2)^{3/2}}$ $\int \frac{\arcsin x dx}{(1+x^2)^{3/2}}$ $\int \frac{\ln(x+\sqrt{x^2-1})dx}{(1+x^2)^{3/2}}$

$\int \frac{\ln x dx}{x^2\sqrt{x^2-1}}$ $\int \frac{\sqrt{1+x^3}dx}{x}$ $\int \frac{x\ln(x+\sqrt{x^2-1})dx}{\sqrt{x^2-1}}$ $\int \frac{x^3\arcsin x dx}{\sqrt{1-x^4}}$

$\int \frac{x^3 arc\sec x dx}{\sqrt{x^4-1}}$ $\int \frac{x\arctan x\ln(x+\sqrt{1+x^2})dx}{\sqrt{1+x^2}}$ $\int \frac{x\ln(1+\sqrt{1-x^2})dx}{\sqrt{1-x^2}}$

$\int \frac{x\ln(x+\sqrt{1+x^2})dx}{\sqrt{1+x^2}}$ $\int \frac{x\ln(x+\sqrt{1-x^2})dx}{\sqrt{1-x^2}}$ $\int \frac{\ln x dx}{x^2\sqrt{1-x^2}}$ $\int \frac{x\arctan x dx}{\sqrt{1+x^2}}$

$\int \frac{\arctan x dx}{x^2\sqrt{1-x^2}}$ $\int \frac{x\arctan x dx}{\sqrt{1-x^2}}$ $\int \frac{\arctan x dx}{x^2\sqrt{1+x^2}}$ $\int \frac{\arcsin x dx}{x^2\sqrt{1-x^2}}$ $\int \frac{x\ln x dx}{\sqrt{x^2-1}}$

$\int \frac{\ln x dx}{x^2\sqrt{1+x^2}}$ $\int \frac{x arc\sec x dx}{\sqrt{x^2-1}}$ $\int \frac{x\ln x dx}{\sqrt{1+x^2}}$ $\int \frac{\sqrt{\sin\theta}d\theta}{1+\sin^2\theta}$ $\int \frac{(1+x^2)dx}{(1-x^2)\sqrt{1+x^4}}$

$\int \frac{(1-x^2)dx}{(1+x^2)\sqrt{1+x^4}}$ $\int \frac{\ln(\sin\theta)d\theta}{1+\sin\theta}$ $\int \ln(\sin\theta)\sqrt{1+\sin\theta}d\theta$ $\int \frac{\sec\theta d\theta}{\sqrt{\sec^4\theta-1}}$

$\int \frac{\tan\theta d\theta}{\sqrt{1+\tan^4\theta}}$ $\int \frac{\sin\theta d\theta}{\sqrt{1-\sin^6\theta}}$ $\int \sqrt{\sqrt{\sec\theta+1}-\sqrt{\sec\theta-1}}d\theta$ $\int x\ln(x^2+1)(\arctan x)^2 dx$

$\int \arctan(x\sqrt{1+x^2})dx$ $\int \arctan(\sqrt{x+1}-\sqrt{x})dx$ $\int \arcsin(x\sqrt{1-x^2})dx$ $\int \arctan(x\sqrt{1-x^2})dx$